

Probability that product of real random matrices have all eigenvalues real tend to 1.

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Abstract

In this article we consider products of real random matrices with fixed size. Let A_1, A_2, \dots be i.i.d $k \times k$ real matrices, whose entries are independent and identically distributed from probability measure μ . Let $X_n = A_1 A_2 \dots A_n$. Then it is conjectured that

$$\mathbb{P}(X_n \text{ has all real eigenvalues}) \rightarrow 1 \text{ as } n \rightarrow \infty.$$

We show that the conjecture is true when μ has an atom.

Introduction and results

The study of products of random matrices was initiated by Furstenberg and Kesten in [5], where they studied the Lyapunov exponents for the product of i.i.d random matrices of fixed size. Since then there has been considerable interest in the study of products of random matrices. For detailed results and applications we refer the reader to [2], [1] and [7]. Not much is known in understanding the spectrum for products of real random matrices.

In [8], Lakshminarayan observed an interesting phenomenon for products of $k \times k$ i.i.d matrices with i.i.d real Gaussian entries in the context of quantum entanglement. Let $p_n^{(k)}$ be the probability that the product of n such matrices has all real eigenvalues. Based on numerical computations, it was conjectured in [8] that $p_n^{(k)}$ increases to 1 with the size of the product.

Forrester, in [3], proved the result by giving an explicit formula for $p_n^{(k)}$, from which it was deduced that this probability increases to 1 exponentially. The same result was shown in [9] following a different approach. We state the a generalization of this result as a conjecture below.

Conjecture 1. *Let X_1, X_2, \dots, X_n be i.i.d matrices of size $k \times k$, whose entries are i.i.d real random variables distributed according to probability measure μ and $A_n = X_1 X_2 \dots X_n$. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_n \text{ has all real eigenvalues}) = 1.$$

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Numerical simulations suggest that the above conjecture is true for any probability measure μ . In [6], numerical evidence was shown for the above conjecture for various probability measures μ , which have continuous density. In [4] a similar computation was done for products of truncated orthogonal matrices.

In this short note we prove the conjecture for a special case when the probability measure μ has an atom i.e., there is a real number x such that $\mu(\{x\}) > 0$. We state the result in Theorem 0.1 and show the special case of the conjecture as a corollary.

Theorem 0.1. *Let X_1, X_2, \dots, X_n be i.i.d random matrices of size $k \times k$, distributed according to a probability measure ν such that $\mathbb{P}(X_1 \text{ has rank } 1) > 0$ and $A_n = X_1 X_2 \dots X_n$. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_n \text{ has all real eigenvalues}) = 1.$$

As a special case if we assume that the random matrices have all entries as i.i.d. random variables distributed according to probability measure μ . If μ has an atom then such random matrices satisfy the above hypothesis and resolve the conjecture 1 when the measure μ has an atom. We state it as the following corollary.

Corollary 0.2. *Let X_1, X_2, \dots, X_n be i.i.d matrices of size $k \times k$, whose entries are i.i.d real random variables distributed according to a probability measure μ which has an atom and $A_n = X_1 X_2 \dots X_n$. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_n \text{ has all real eigenvalues}) = 1.$$

Any probability measure μ with discrete support satisfy the hypothesis of the above theorem. For example if the entries of the matrices are Rademacher distributed random variables (takes values ± 1 with equal probability), then the theorem asserts that product of such matrices have purely real spectrum with exponentially high probability.

The proof of the Theorem 0.1 is elementary and based on a simple observation that rank of product of matrices is at most the minimum of the ranks of the individual matrices. In the given scenario, each individual matrix will be of rank at most 1 with non zero probability. If a real matrix has rank at most 1, then it has all real eigenvalues (they are 0 and the trace of the matrix).

Proof of Theorem 0.1.

$$\begin{aligned} \mathbb{P}(A_n \text{ has rank at most } 1) &\geq \mathbb{P}(\text{at least one of } X_1, X_2, \dots, X_n \text{ has rank at most } 1), \\ &\geq 1 - (1 - \mathbb{P}(X_1 \text{ has rank } 1))^n. \end{aligned} \tag{0.1}$$

We know that real matrices with rank at most 1, have all eigenvalues real. Hence,

$$\mathbb{P}(A_n \text{ has all real eigenvalues}) \geq \mathbb{P}(A_n \text{ has rank at most } 1).$$

Therefore from above and (0.1) we have,

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_n \text{ has all real eigenvalues}) = 1.$$

□

We now show a lower bound, that is away from 0, for the probability that the product of 2×2 i.i.d real random matrices having both the real eigenvalues.

Proposition 0.3. *Let X_1, X_2, \dots, X_n are i.i.d matrices of size 2×2 whose entries are i.i.d real random variables distributed according to probability measure μ and $A_n = X_1 X_2 \dots X_n$. Then,*

$$\mathbb{P}(A_n \text{ has all real eigenvalues}) \geq \frac{1}{2}.$$

Notice that the rows of product of i.i.d random matrices are exchangeable. We show the bound for the matrices whose rows are exchangeable in the following Lemma 0.4. Hence the proof of Proposition 0.3 follows from the Lemma 0.4.

Lemma 0.4. *Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where (a, b) and (c, d) are real exchangeable random variables. Then,*

$$\mathbb{P}(M \text{ has both real eigenvalues}) \geq \frac{1}{2}.$$

Proof. The characteristic polynomial of the matrix M is $P_M(x) = x^2 - (a+d)x + (ad - bc)$. The matrix M has all real eigenvalues if and only if the discriminant of the characteristic polynomial,

$$(a + d)^2 - 4(ad - bc) \geq 0.$$

Because (a, c) and (b, d) are exchangeable we have,

$$\mathbb{P}((a + d)^2 - 4(ad - bc) \geq 0) = \mathbb{P}((b + c)^2 - 4(bc - bd) \geq 0).$$

Therefore,

$$\begin{aligned} \mathbb{P}(M \text{ has both real eigenvalues}) &= \frac{1}{2}(\mathbb{P}((a + d)^2 - 4(ad - bc) \geq 0) + \mathbb{P}((b + c)^2 - 4(bc - bd) \geq 0)), \\ &\geq \frac{1}{2}\mathbb{P}((a + d)^2 - 4(ad - bc) \geq 0 \text{ or } (b + c)^2 - 4(bc - bd) \geq 0). \end{aligned} \tag{0.2}$$

Because $(a + d)^2 - 4(ad - bc) + (b + c)^2 - 4(bc - bd) \geq 0$, at least one of $(a + d)^2 - 4(ad - bc)$ and $(b + c)^2 - 4(bc - bd)$ is non-negative. Therefore,

$$\mathbb{P}((a + d)^2 - 4(ad - bc) \geq 0 \text{ or } (b + c)^2 - 4(bc - bd) \geq 0) = 1.$$

Combining above and (0.2) we have that, $\mathbb{P}(M \text{ has both real eigenvalues}) \geq \frac{1}{2}$. \square

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References

- [1] P. Bougerol and J. Lacroix, *Products of random matrices with applications to Schrödinger operators*, ch. The Pure Point Spectrum, pp. 237–251, Birkhäuser Boston, Boston, MA, 1985.
- [2] J.E. Cohen, H. Kesten, and C.M. Newman, *Random matrices and their applications: Proceedings of the AMS-IMS-SIAM Joint summer research conference June 17-23, 1984*, Contemporary mathematics, American Mathematical Society, 1986.
- [3] Peter J Forrester, *Probability of all eigenvalues real for products of standard gaussian matrices*, Journal of Physics A: Mathematical and Theoretical **47** (2014), no. 6, 065202.
- [4] P.J. Forrester and S. Kumar, *The probability that all eigenvalues are real for products of truncated real orthogonal random matrices*, arXiv:1606.03670 [math-ph] (2016).
- [5] H. Furstenberg and H. Kesten, *Products of random matrices*, Ann. Math. Statist. **31** (1960), 457–469. MR 0121828
- [6] S. Hameed, K. Jain, and A. Lakshminarayan, *Real eigenvalues of non-gaussian random matrices and their products*, Journal of Physics A: Mathematical and Theoretical **48** (2015), no. 38, 385204.
- [7] G. Högnäs and A. Mukherjea, *Probability measures on semigroups: Convolution products, random walks and random matrices*, Probability and Its Applications, Springer US, 2010.
- [8] Arul Lakshminarayan, *On the number of real eigenvalues of products of random matrices and an application to quantum entanglement*, Journal of Physics A: Mathematical and Theoretical **46** (2013), no. 15, 152003.
- [9] Nanda Kishore Reddy, *Equality of Lyapunov and stability exponents for products of isotropic random matrices*, arXiv:1601.02888 [math.PR] (2016).
- [10] Tulasi Ram Reddy, *On critical points of random polynomials and spectrum of certain products of random matrices*, Ph.D. thesis, Indian Institute of Science, Bangalore, 2016, arXiv:1602.05298 [math.PR].